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Reconstruction of the Clarke Subdifferential by the Lasry–Lions Regularizations[1]

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We prove that the Clarke subdifferential of a locally Lipschitz function with a growth condition, defined on a Hilbert space, can be represented by the derivatives of its Lasry–Lions regularizations. We complete a result of J. Benoist, showing a similar representation for the Clarke subdifferential of a Lipschitz function in an arbitrary Banach space, by the Clarke subdifferentials of its Lasry–Lions regularizations. © 2000 Academic Press

1. INTRODUCTION

The Lasry–Lions regularizations, introduced in [8] and developed in [1], are useful tools for approximation of real valued functions in Hilbert spaces. One of their good features is that they are of class $C^{1,1}(H)$. In this connection a natural question arises: What is the relationship between their derivatives and the subdifferentials of the approximated functions? In

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this paper we answer this question showing that the Clarke subdifferential of a locally Lipschitz function with a growth condition can be represented by the derivatives of its Lasry–Lions regularizations.

Similar questions about the relationship between the Clarke subdifferential of a Lipschitz function in normed linear space and the Clarke subdifferentials of its Lasry–Lions regularizations have already been considered by J. Benoist [2]. Here we complete his result, proving the inverse inclusion in his main theorem (see [2, Theorem 3]).

Other results in this paper show that the Moreau–Yosida regularizations of a lower semicontinuous function with a growth condition are Fréchet differentiable almost everywhere (in the Baire sense) and have some rudimentary properties (in a comparison with the convex case) with respect to the Fréchet subdifferentials of the approximated function.

Some of the results in this paper were announced in [7].

2. PRELIMINARIES

We recall some properties of the Moreau–Yosida and the Lasry–Lions regularizations for an illustration of their important features.

Let $(H, \|.\|)$ be a Hilbert space, S be its unit sphere, B^o (resp. B) be its open (resp. closed) unit ball, and $B(x, r)$ (resp. $B[x, r]$) denote the open (resp. closed) ball with center x and radius r. Let the extended real valued functions $f : H \to \mathbb{R} \cup \{+\infty\}$ and $g : H \to \mathbb{R} \cup \{-\infty\}$ be given.

The *Moreau–Yosida epigraphical regularization* of index $\lambda > 0$ of f is defined by

$$f_\lambda(x) := \inf_{y \in H} \left\{ f(y) + \frac{1}{2\lambda}\|x - y\|^2 \right\}.$$

It is natural to consider the symmetric notion of hypographical regularization for g.

The *Moreau–Yosida hypographical regularization* of index $\mu > 0$ of g is defined by

$$g^\mu(x) := \sup_{y \in H} \left\{ g(y) - \frac{1}{2\mu}\|x - y\|^2 \right\}.$$

Assume that there exist constants $c, d > 0$ such that for all $x \in H$

$$f(x) \geq -\frac{c}{2}(\|x\|^2 + 1), \tag{2.1}$$

$$g(x) \leq \frac{d}{2}(\|x\|^2 + 1).$$

Then (see [1, Proposition 1.1]), provided $\lambda \in (0, 1/c)$ (resp. $\mu \in (0, 1/d)$), the function f_λ (resp. g^μ) is a finitely valued function and Lipschitz continuous on each bounded subset of H. Moreover for all $x \in H$

$$\sup_{\lambda > 0} f_\lambda(x) = \underline{cl} f(x), \qquad \inf_{\mu > o} g^\mu(x) = \overline{cl} g(x),$$

where $\underline{cl} f(x)$ is the lower semicontinuous regularization of f, namely $\underline{cl} f(x) = \sup_{\varepsilon > 0} \inf_{x \in B(x, \varepsilon)} f(z)$ and $\overline{cl} g(x)$ is the upper semicontinuous regularization of g, i.e., $\overline{cl} g(x) = \inf_{\varepsilon > 0} \sup_{x \in B(x, \varepsilon)} g(z)$.

Despite the global definition, the regularization operation has a local character. The proof of [1, Proposition 1.2(a)] shows that if $f(x)$ is finite, for $\lambda \in (0, 1/2c)$

$$f_\lambda(x) < \inf_{\|x - y\| > \rho} \left\{ f(y) + \frac{1}{2\lambda} \|x - y\|^2 \right\}, \tag{2.2}$$

where ρ is given by

$$\rho = \rho(x, f, \lambda, c) = \left[\lambda \frac{2f(x) + c(2\|x\|^2 + 1)}{1 - 2\lambda c} \right]^{1/2}. \tag{2.3}$$

Similarly, the hypographical regularization also has a local character. If $g(x)$ is finite, for $\mu \in (0, 1/2d)$

$$g^\mu(x) > \sup_{\|x - y\| > \sigma} \left\{ g(y) - \frac{1}{2\mu} \|x - y\|^2 \right\},$$

where σ is given by

$$\sigma = \sigma(x, g, \mu, d) = \left[\mu \frac{d(2\|x\|^2 + 1) - 2g(x)}{1 - 2\mu d} \right]^{1/2}.$$

A function $f : H \to \mathbb{R} \cup \{+\infty\}$ is said to be *weakly convex*, or convex up to a square, if there exists some constant $c > 0$ such that $f(.) + \frac{c}{2}\|.\|^2$ is convex. This property is denoted by $f \in \Gamma_c(H)$. A function $g : H \to \mathbb{R} \cup \{-\infty\}$ is said to be *weakly concave* if $-g$ is weakly convex. This is equivalent to the existence of some constant $c > 0$ such that $g(.) - \frac{c}{2}\|.\|^2$ is concave.

For any $\lambda \in (0, 1/c)$, $-f_\lambda \in \Gamma_{1/\lambda}(H)$ and for any $\mu \in (0, 1/d)$, $g^\mu \in \Gamma_{1/\mu}(H)$ (see [1, Proposition 3.2]).

J.-M Lasry and P.-L. Lions [8] introduced the functions

$$(f_\lambda)^\mu(x) = \sup_{y\in H} \inf_{z\in H} \left\{ f(z) + \frac{1}{2\lambda}\|z-y\|^2 - \frac{1}{2\mu}\|y-x\|^2 \right\},$$

$$(g^\lambda)_\mu(x) = \inf_{y\in H} \sup_{z\in H} \left\{ g(z) - \frac{1}{2\lambda}\|z-y\|^2 + \frac{1}{2\mu}\|y-x\|^2 \right\}.$$

For all $0 < \mu < \lambda < 1/c$ (resp. $0 < \mu < \lambda < 1/d$), $(f_\lambda)^\mu$ (resp. $(g^\lambda)_\mu$) is the $C^{1,1}(H)$ function whose gradient is $\max\{\frac{1}{\mu}, \frac{1}{\lambda-\mu}\}$-Lipschitz continuous (see [1, Theorem 4.1]). The function $(f_\lambda)^\mu \in \Gamma_{1/\mu}(H)$ and $-(f_\lambda)^\mu \in \Gamma_{1/(\lambda-\mu)}(H)$. Moreover, $(f_\lambda)^\mu \le f$, $(g^\lambda)_\mu \ge g$, and

$$\lim_{\substack{\lambda\to 0\\ \mu\to 0\\ \lambda>\mu}} (f_\lambda)^\mu(x) = \underline{cl}f(x), \qquad \lim_{\substack{\lambda\to 0\\ \mu\to 0\\ \lambda>\mu}} (g^\lambda)_\mu(x) = \overline{cl}g(x).$$

We shall give some more definitions and results, which will be used in the sequel.

Recall that the *Fréchet subdifferential* of a function $f: H \to \mathbb{R} \cup \{+\infty\}$ at x is the set

$$\partial^F f(x) := \left\{ x^* \in H : \liminf_{\|h\|\to 0} \frac{f(x+h) - f(x) - \langle x^*, h\rangle}{\|h\|} \ge 0 \right\}.$$

It is known (see [4]) that for a lower semicontinuous function f the set $\operatorname{Dom} \partial^F f = \{x : \partial^F(x) \ne \emptyset\}$ is dense in the set $\operatorname{dom} f = \{x : f(x) < +\infty\}$. The point x is said to be a *point of Fréchet subdifferentiability* of f, if $x \in \operatorname{Dom} \partial^F f$.

It is easy to see that $\partial^F f(x) \subset \partial f(x)$ for every x for a locally Lipschitz function f, where ∂f stands for the Clarke subdifferential of f. Recall that the *Clarke subdifferential* of a locally Lipschitz function f at the point x is the set

$$\partial f(x) = \{ p \in H : f^0(x; h) \ge \langle p, h\rangle, \forall h \in H \}.$$

where

$$f^0(x;h) = \limsup_{\substack{y\to x\\ t\downarrow 0}} \frac{f(y+th) - f(y)}{t}.$$

3. DIFFERENTIABILITY PROPERTIES OF MOREAU–YOSIDA APPROXIMATIONS OF LOWER SEMICONTINUOUS FUNCTIONS

In this section we show that the Moreau–Yosida approximations of a proper lower semicontinuous function have some rudimentary properties (in comparison with the convex case) with respect to the Fréchet subdifferential of the approximated function.

THEOREM 3.1. *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, such that* (2.1) *is satisfied. Then for every $\lambda \in (0, 1/2c)$ we have*:

(a) *at every point x of Fréchet subdifferentiability of f_λ there exists $y_\lambda(x)$, such that*

$$\|y_\lambda(x) - x\| \leq \rho(x, f, \lambda, c), \tag{3.1}$$

where $\rho(x, f, \lambda, c)$ is given by (2.3),

$$f_\lambda(x) = f(y_\lambda(x)) + \frac{1}{2\lambda}\|x - y_\lambda(x)\|^2, \tag{3.2}$$

and

$$\partial^F(f_\lambda)(x) \subset \partial^F f(y_\lambda(x)); \tag{3.3}$$

(b) *the set G_λ on which f_λ is Fréchet differentiable is dense G_δ. Moreover, if $x \in G_\lambda$, then the point $y_\lambda(x)$ is the point of strong minimum of the function $\varphi_x(.) := f(.) + \frac{1}{2\lambda}\|x - .\|^2$ (i.e., every minimizing sequence for φ_x converges to $y_\lambda(x)$), $y_\lambda : G_\lambda \to H$ is a continuous map, and*

$$(f_\lambda)'(x) = \frac{x - y_\lambda(x)}{\lambda}; \tag{3.4}$$

(c) *the range of the operator $(I + \lambda\,\partial^F f) : H \to H$ contains G_λ*;

(d) *the resolvent $J_\lambda = (I + \lambda\,\partial^F f)^{-1}$ is non-empty on G_λ and for every $x \in G_\lambda$ we have that $J_\lambda(x) \ni y_\lambda(x)$.*

Proof. To establish (a) we apply [3, Theorem 11] and obtain that there exists a point $y_\lambda(x)$, such that (3.2) is fulfilled for any point x of Fréchet subdifferentiability of f_λ. It is easy to see that for any $h \in H$

$$f(y_\lambda(x) + h) - f(y_\lambda(x)) \geq f_\lambda(x + h) - f_\lambda(x).$$

To establish (3.3) take an arbitrary $p \in \partial^F f_\lambda(x)$. Then

$$\begin{aligned} &\liminf_{\|h\| \to 0} \frac{f(y_\lambda(x) + h) - f(y_\lambda(x)) - \langle p, h\rangle}{\|h\|} \\ &\qquad \geq \lim_{\|h\| \to 0} \frac{f_\lambda(x + h) - f_\lambda(x) - \langle p, h\rangle}{\|h\|} \geq 0, \end{aligned}$$

so

$$p \in \partial^F f(y_\lambda(x)).$$

It remains to observe that (3.1) follows from (2.2).

Since $\lambda \in (0, 1/2c)$ and (2.1) is satisfied, we can apply Proposition 1.1 from [1] to obtain that f_λ is finite valued and Lipschitz continuous on each bounded subset of H. Now (b) follows from the Ekeland–Lebourg theorem (see [6, Theorem 2.5]), where its conditions are satisfied thanks to (2.1) and the simple fact that the set of Fréchet differentiability of a convex function is always a G_δ set.

(c) This follows from (3.4).

(d) $y \in J_\lambda(x)$ if and only if $x \in (I + \lambda\, \partial^F f)(y)$ and the assertion follows from (c). ∎

As an application of (3.3) and the Ekeland–Lebourg theorem we give the following

PROPOSITION 3.2. *Let $f : H \to \mathbb{R}$ be a lower semicontinuous, bounded below and coercive function, i.e., such that $\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = +\infty$. Then the range of $\partial^F f$ contains a dense G_δ set.*

Proof. Put $g(x) = f(x) - \frac{1}{2}\|x\|^2$ and consider

$$g_1(x) = \inf_{y \in H} \left\{ g(y) + \tfrac{1}{2}\|x - y\|^2 \right\} = \inf_{y \in H} \left\{ f(y) - \langle x, y \rangle + \tfrac{1}{2}\|x\|^2 \right\}.$$

From the coercivity of f we have that the level sets of the function in brackets are norm bounded and the Ekeland–Lebourg theorem [6, Theorem 2.5] gives that there exists a dense and G_δ set of Fréchet differentiability of g_1 and for x from this set, using (3.3) we have that

$$x - y_1(x) = (g_1)'(x) \in \partial^F g(y_1(x)) \subset \partial^F f(y_1(x)) - y_1(x),$$

i.e.,

$$x \in \partial^F f(y_1(x)).$$

∎

4. RECONSTRUCTION OF THE CLARKE SUBDIFFERENTIAL OF A LOCALLY LIPSCHITZ FUNCTION

In this section we prove that if f is a locally Lipschitz function and x^* is a weak limit of the derivatives $((f_{\lambda_n})^{\mu_n})'(x_n)$ for some sequences $x_n \xrightarrow[n \to \infty]{} x$, $\lambda_n \xrightarrow[n \to \infty]{} 0$, $\mu_n/\lambda_n \xrightarrow[n \to \infty]{} 0$, then $x^* \in \partial f(x)$. Moreover, $\partial f(x)$ is the closed convex hull of all such limits.

Having in mind that if $f : H \to \mathbb{R}$ is a lower semicontinuous function, satisfying the growth condition (2.1), then for $\lambda \in (0, 1/2c)$ the function f_λ is locally Lipschitz and $-(f_\lambda)^\mu(x) = (-f_\lambda)_\mu(x)$ is of class $C^{1,1}(H)$ for

$\mu < \lambda$. By Theorem 3.1 we have that for all $x \in H$ there exists an unique point $y_{\lambda\mu}(x)$, such that

$$\big((f_\lambda)^\mu\big)'(x) = \frac{y_{\lambda\mu}(x) - x}{\mu}$$

and $-((f_\lambda)^\mu)'(x) \in \partial^F(-f_\lambda)(y_{\lambda\mu}(x)) \subset \partial(-f_\lambda)(y_{\lambda_\mu}(x))$, i.e.,

$$\big((f_\lambda)^\mu\big)'(x) \in \partial f_\lambda\big(y_{\lambda\mu}(x)\big).$$

It is easy to see that, in this case, for sufficiently small λ the function f_λ is quadratically majorized with a constant $d_\lambda = \frac{\ell}{\lambda}$, where ℓ is a positive constant, bounded on bounded sets, $\|y_{\lambda\mu}(x) - x\| \leq \sigma(x, f_\lambda, \mu, d_\lambda)$ and then $y_{\lambda\mu}(x) \to x$, whenever $\frac{\mu}{\lambda}$ tends to zero.

Moreover, when f is locally Lipschitz and satisfies (2.1), then it is bounded on a bounded neighborhood $U(x_0)$ of x_0, on which it is Lipschitz and $\sup_{x \in U(x_0)} \rho(x, f, \lambda, c)$ tends to 0, as well as λ tends to 0. When the ratio $\frac{\mu}{\lambda}$ tends to 0, $\sup_{x \in U(x_0)} \sigma(x, f_\lambda, \mu, d_\lambda)$ also tends to 0. These observations will be used in the proofs below.

Let us recall, as well, the well known theorem of D. Preiss (see [9]), which gives a representation of the Clarke subdifferential of a locally Lipschitz function $f : H \to \mathbb{R}$ by its Fréchet derivatives,

$$\partial f(x) = \bigcap_{s>0} \overline{\mathrm{co}}\{f'(y) : y \in B(x,s)\},$$

and the mean-value inequality

$$\inf_{z \in V} \langle f'(z), u - v\rangle \leq f(u) - f(v) \leq \sup_{z \in V} \langle f'(z), u - v\rangle, \quad (4.1)$$

for every $u, v \in \operatorname{dom} f$, where V is an arbitrary open set, containing $[u, v]$.

DEFINITION 4.1.

$$\limsup_{\substack{\lambda \to 0 \\ \frac{\mu}{\lambda} \to 0 \\ z \to x}} \big((f_\lambda)^\mu\big)'(z) := \Big\{ x^* \in H : x^* = w - \lim_{n \to \infty} \Big((f_{\lambda_n})^{\mu_n}\Big)'(x_n), \quad \lambda_n \xrightarrow[n\to\infty]{} 0,\ \frac{\mu_n}{\lambda_n} \xrightarrow[n\to\infty]{} 0,\ x_n \xrightarrow[n\to\infty]{} x \Big\}.$$

THEOREM 4.2. *Let $f : H \to \mathbb{R}$ be a locally Lipschitz function that satisfies the growth condition* (2.1). *Then*

$$\overline{\mathrm{co}} \limsup_{\substack{\lambda \to 0 \\ \frac{\mu}{\lambda} \to 0 \\ z \to x}} \big((f_\lambda)^\mu\big)'(z) = \partial f(x).$$

Proof. Denote

$$A(x) = \limsup_{\substack{\lambda \to 0 \\ \frac{\mu}{\lambda} \to 0 \\ z \to x}} \big((f_\lambda)^\mu \big)'(z).$$

Then $\overline{\mathrm{co}}\, A(x)$ is a closed convex set. Suppose that there exists $\rho \in \partial f(x)$ such that $p \notin \overline{\mathrm{co}}\, A(x)$. Then by the separation theorem there exist $h \in S$ and $\delta > 0$ such that

$$\langle p, h \rangle \geq \sup_{a \in A(x)} \langle a, h \rangle + \delta. \tag{4.2}$$

Let L stand for the Lipschitz constant of f on a bounded neighborhood $U(x)$. By the definition of Clarke's directional derivative there exist sequences $x_n \xrightarrow[n \to \infty]{} x$, and $t_n \downarrow 0$, such that

$$f^0(x, h) = \limsup_{\substack{z \to x \\ t \downarrow 0}} \frac{f(z + th) - f(z)}{t} = \lim_{n \to \infty} \frac{f(x_n + t_n h) - f(x_n)}{t_n}.$$

For sufficiently large $n \in \mathbb{N}$ the points x_n and $x_n + t_n h$ are in $U(x)$ and we can take $\lambda_n > 0$ such that $\sup_{z \in U(x)} \rho(z, f, \lambda_n, c) < t_n^2$. Having in mind the local Lipschitz continuity of f and density of the sets G_{λ_n} (recall that G_λ is given by Theorem 3.1) we may without loss of generality assume that $x_n + t_n h \in G_{\lambda_n}$, so there exists $y_{\lambda_n}(x_n + t_n h)$, such that

$$f_{\lambda_n}(x_n + t_n h) = f\big(y_{\lambda_n}(x_n + t_n h)\big) + \frac{1}{2\lambda_n} \big\| x_n + t_n h - y_{\lambda_n}(x_n + t_n h) \big\|^2,$$

$$(f_{\lambda_n})'(x_n + t_n h) \in \partial f\big(y_{\lambda_n}(x_n + t_n h)\big),$$

$$\big\| x_n + t_n h - y_{\lambda_n}(x_n + t_n h) \big\| \leq \rho(x_n + t_n h, f, \lambda_n, c) \leq t_n^2.$$

Here we conclude that

$$\begin{aligned} f^0(x, h) &= \lim_{n \to \infty} \frac{f\big(y_{\lambda_n}(x_n + t_n h)\big) - f\big(y_{\lambda_n}(x_n + t_n h) - t_n h\big)}{t_n} \\ &\leq \limsup_{n \to \infty} \frac{f_{\lambda_n}(x_n + t_n h) - f_{\lambda_n}(x_n)}{t_n}. \end{aligned} \tag{4.3}$$

Using the fact that f_{λ_n} is a Lipschitz function on bounded sets, for every $n \in \mathbb{N}$ we can take $\mu_n > 0$, such that $\mu_n / \lambda_n < 1/n$ and $\sup_{z \in U(x)} \sigma(z, f_{\lambda_n}, \mu_n, c) < \min\{t_n^2 / 2L_{\lambda_n}, 1/n\}$, where L_{λ_n} is the Lipschitz constant of f_{λ_n} on $U(x)$. Since $(f_{\lambda_n})^{\mu_n} \in C^{1,1}(H)$, by Theorem 3.1 we have that there exists $y_{\lambda_n\mu_n}(x_n)$, such that

$$(f_{\lambda_n})^{\mu_n}(x_n) = f_{\lambda_n}\big(y_{\lambda_n\mu_n}(x_n)\big) - \frac{1}{2\mu_n}\left\| x_n - y_{\lambda_n\mu_n}(x_n)\right\|^2,$$

$$\Big((f_{\lambda_n})^{\mu_n}\Big)'(x_n) \in \partial f_{\lambda_n}\big(y_{\lambda_n\mu_n}(x_n)\big),$$

$$\left\| x_n - y_{\lambda_n\mu_n}(x_n)\right\| \le \sigma\big(x_n, f_{\lambda_n}, \mu_n, d_{\lambda_n}\big) < \frac{t_n^2}{2L_{\lambda_n}}.$$

From the choice of μ_n and the Lipschitz continuity of f_{λ_n} it follows that

$$\begin{aligned}
&f_{\lambda_n}(x_n + t_n h) - f_{\lambda_n}(x_n)\\
&\quad \le f_{\lambda_n}\big(y_{\lambda_n\mu_n}(x_n) + t_n h\big) - f_{\lambda_n}\big(y_{\lambda_n\mu_n}(x_n)\big) + 2L_{\lambda_n}\left\| x_n - y_{\lambda_n\mu_n}(x_n)\right\|\\
&\quad \le f_{\lambda_n}\big(y_{\lambda_n\mu_n}(x_n) + t_n h\big) - f_{\lambda_n}\big(y_{\lambda_n\mu_n}(x_n)\big) + t_n^2\\
&\quad \le (f_{\lambda_n})^{\mu_n}(x_n + t_n h) + \frac{1}{2\mu}\left\| x_n - y_{\lambda_n\mu_n}(x_n)\right\|^2 - f_{\lambda_n}\big(y_{\lambda_n\mu_n}(x_n)\big) + t_n^2\\
&\quad = (f_{\lambda_n})^{\mu_n}(x_n + t_n h) - (f_{\lambda_n})^{\mu_n}(x_n) + t_n^2\\
&\quad = \Big\langle \Big((f_{\lambda_n})^{\mu_n}\Big)'(w_n), t_n h\Big\rangle + t_n^2,
\end{aligned}$$

where $w_n \in (x_n, x_n + t_n h)$ by the mean-value theorem. By Theorem 3.1 we have that $((f_{\lambda_n})^{\mu_n})'(w_n) \in \partial f_{\lambda_n}(y_{\lambda_n\mu_n}(w_n))$ and by the Preiss theorem it follows that $((f_{\lambda_n})^{\mu_n})'(w_n) \in \overline{\mathrm{co}}\{(f_{\lambda_n})'(y) : y \in G_{\lambda_n} \cap B(y_{\lambda_n\mu_n}(w_n), s)\}$ for every $s > 0$. Again by Theorem 3.1, $(f_{\lambda_n})'(y) \in \partial f(y_{\lambda_n}(y))$ and the latter set is contained in LB for small s and large n, having in mind also the choice of λ_n and μ_n. So we proved that for sufficiently large n the derivatives $((f_{\lambda_n})^{\mu_n})'(w_n)$ are norm bounded, so we can extract from them a weakly converging subsequence, whose limit is $a_0 \in A(x)$. Hence, using (4.3) we have that

$$\langle p, h\rangle \le f^0(x, h) \le \langle a_0, h\rangle.$$

This yields a contradiction with (4.2), hence

$$\overline{\mathrm{co}}\, A(x) \supset \partial f(x). \tag{4.4}$$

To show the opposite inclusion take arbitrary $q \in A(x)$, i.e., $q = w-\lim((f_{\lambda_n})^{\mu_n})'(z_n)$, where $z_n \xrightarrow[n\to\infty]{} x$, $\mu_n/\lambda_n \downarrow 0$, $\lambda_n \downarrow 0$. Assume that $q \notin$

$\partial f(x)$. By the separation theorem there exist $h \in S$, $s > 0$ such that

$$\langle q, h \rangle \geq \sup_{p \in \partial f(x)} \langle p, h \rangle + s.$$

From the norm to weak upper semicontinuity of ∂f there exists $\varepsilon > 0$ such that for sufficiently small μ_n and λ_n and z_n sufficiently close to x

$$\left\langle \left((f_{\lambda_n})^{\mu_n} \right)'(z_n), h \right\rangle \geq \sup_{p \in \partial f(x)} \langle p, h \rangle + \frac{3s}{4} \geq \sup_{d \in D_\varepsilon(x)} \langle d, h \rangle + \frac{s}{2}, \quad (4.5)$$

where $D_\varepsilon(x) = \overline{\mathrm{co}}\{d \in \partial f(z),\ z \in B(x, \varepsilon) \subset U(x)\}$. From Theorem 3.1 there exist $y_{\lambda_n \mu_n}(z_n)$, such that

$$\left((f_{\lambda_n})^{\mu_n} \right)'(z_n) \in \partial f_{\lambda_n}\left(y_{\lambda_n \mu_n}(z_n) \right),$$
$$\left\| z_n - y_{\lambda_n \mu_n}(z_n) \right\| \leq \sigma\left(z_n, f_{\lambda_n}, \mu_n, d_{\lambda_n} \right).$$

Using again the Preiss representation of ∂f_{λ_n}, we have that $((f_{\lambda_n})^{\mu_n})'(z_n) \in \overline{\mathrm{co}}\{(f_{\lambda_n})'(y) : y \in G_{\lambda_n} \cap B(y_{\lambda_n \mu_n}(z_n), s)\}$ for every $s > 0$ and again by Theorem 3.1 we have that $(f_{\lambda_n})'(y) \in \partial f(y_{\lambda_n}(y))$, where $\|y - y_{\lambda_n}(y)\| \leq \rho(y, f, \lambda_n, c)$. For sufficiently small s, having in mind the estimations we have that for sufficiently small λ_n, μ_n / λ_n and z_n close to x, the points $y_{\lambda_n}(y) \in B(x, \varepsilon) \subset U(x)$ and the set $\partial f(y_{\lambda_n}(y)) \subset D_\varepsilon(x)$. So for large $n \in \mathbb{N}$ and small $s > 0$ by (4.5) we obtain a contradiction. Hence

$$\overline{\mathrm{co}}\, A(x) \subset \partial f(x)$$

and combining with (4.4) we complete the proof. ∎

For the next definition we recall that G_λ is defined in Theorem 3.1(b).

DEFINITION 4.3.

$$\limsup_{\substack{G_\lambda \ni z \to x \\ \lambda \downarrow 0}} (f_\lambda)'(z) := \Big\{ x^* \in H : x^* = w - \lim (f_{\lambda_n})'(x_n),$$
$$\lambda_n \xrightarrow[n \to \infty]{} 0,\ x_n \in G_{\lambda_n},\ x_n \xrightarrow[n \to \infty]{} x \Big\}.$$

THEOREM 4.4. *Let $f : H \to \mathbb{R}$ be a locally Lipschitz function that satisfies the growth condition* (2.1). *Then*

$$\overline{\mathrm{co}} \limsup_{\substack{G_\lambda \ni z \to x \\ \lambda \downarrow 0}} (f_\lambda)'(z) = \partial f(x).$$

Proof. Denote

$$C(x) = \limsup_{\substack{G_\lambda \ni z \to x \\ \lambda \downarrow 0}} (f_\lambda)'(z).$$

Then $\overline{\mathrm{co}}\, C(x)$ is a closed convex set. Suppose that there exists $p \in \partial f(x)$ such that $p \notin \overline{\mathrm{co}}\, C(x)$. Then by the separation theorem there exist $h \in S$ and $\delta > 0$ such that

$$\langle p, h \rangle \geq \sup_{c \in C(x)} \langle c, h \rangle + \delta. \tag{4.6}$$

Following step by step the proof of Theorem 4.2 until (4.3), we have, using the mean-value inequality (4.1) of Preiss, that

$$\begin{aligned}
&f\big(y_{\lambda_n}(x_n + t_n h)\big) - f\big(y_{\lambda_n}(x_n + t_n h) - t_n h\big) \\
&\quad \leq f_{\lambda_n}(x_n + t_n h) - f_{\lambda_n}(x_n) \\
&\quad \leq \sup_{z \in V_n \cap G_{\lambda_n}} \big\langle (f_{\lambda_n})'(z), t_n h \big\rangle \\
&\quad \leq \big\langle (f_{\lambda_n})'(z_n), t_n h \big\rangle + \frac{\delta}{2},
\end{aligned}$$

where $V_n = [x_n, x_n + t_n h] + \frac{1}{n} B^o$ and $z_n \in V_n \cap G_{\lambda_n}$. For sufficiently large $n \in \mathbb{N}$ we have that $y_{\lambda_n}(z_n) \in U(x)$, where by definition f is Lipschitz. So by Theorem 3.1 there exists $N \in \mathbb{N}$, such that the set of the derivatives $\{(f_{\lambda_n})'(z_n)\}_{n \geq N}$ is norm bounded. We can extract from them a weakly convergened subsequence with limit $c_0 \in C(x)$. Therefore by (4.3),

$$\langle p, h \rangle \leq f^0(x, h) \leq \langle c_0, h \rangle + \frac{\delta}{2},$$

which is a contradiction with (4.6), hence

$$\overline{\mathrm{co}}\, C(x) \supset \partial f(x). \tag{4.7}$$

To show the opposite inclusion take arbitrary $q \in C(x)$, i.e., $q = w - \lim (f_{\lambda_n})'(z_n)$, where $G_{\lambda_n} \ni z_n \xrightarrow[n \to \infty]{} x$, $\lambda_n \xrightarrow[n \to \infty]{} 0$. Assume that $q \notin \partial f(x)$. By the separation theorem there exist $h \in S$, $s > 0$ such that

$$\langle q, h \rangle \geq \sup_{p \in \partial f(x)} \langle p, h \rangle + s.$$

From the norm to weak upper semicontinuity of ∂f there exists $\varepsilon > 0$ such that for sufficiently small $\lambda_n > 0$ and z_n close to x

$$\left\langle \left(f_{\lambda_n}\right)'(z_n), h\right\rangle \geq \sup_{p \in \partial f(x)} \langle p, h\rangle + \frac{3s}{4} \geq \sup_{d \in D_\varepsilon(x)} \langle d, h\rangle + \frac{s}{2}, \quad (4.8)$$

where $D_\varepsilon(x)$ is defined in (4.5). From Theorem 3.1 there exist $y_{\lambda_n}(z_n)$, such that

$$\left(f_{\lambda_n}\right)'(z_n) \in \partial f\left(y_{\lambda_n}(z_n)\right),$$

$$\left\| z_n - y_{\lambda_n}(z_n)\right\| \leq \rho(z_n, f, \lambda_n, c).$$

It is clear that for sufficiently small $\lambda_n > 0$ and z_n close to x the points $y_{\lambda_n}(z_n) \in B(x, \varepsilon) \subset U(x)$ and $(f_{\lambda_n})'(z_n) \in D_\varepsilon(x)$. Hence for large n we obtain a contradiction with (4.8); therefore

$$\overline{\text{co}}\, C(x) \subset \partial f(x)$$

and this combined with (4.7) completes the proof. ∎

5. REPRESENTATION OF THE CLARKE SUBDIFFERENTIAL OF A LIPSCHITZ FUNCTION BY THE CLARKE SUBDIFFERENTIALS OF ITS LASRY–LIONS REGULARIZATIONS

In this section we complete a result of J. Benoist, showing that the Clarke subdifferential of a Lipschitz function, defined on an arbitrary Banach space, can be characterized, roughly speaking, as a w^*-upper semicontinuous envelope of the Clarke subdifferentials of its Lasry–Lions regularizations. Let us remark that the proof of this result uses essentially the global Lipschitz property of the considered function.

THEOREM 5.1. *Let $(E, \|.\|)$ be a Banach space and $f: E \to \mathbb{R}$ be a Lipschitz function. Then*

$$\overline{\text{co}}^*\, L(f, x) = \partial f(x),$$

where

$$L(f, x) = \bigcap_{\substack{\varepsilon > 0 \\ \nu > 0}} \overline{\bigcup_{\substack{y \in B(x;\, \varepsilon) \\ 0 < \mu < \lambda < \nu}} \partial (f_\lambda)^\mu(y)}^{w^*}.$$

Proof. The inclusion $\subset$ is proved by J. Benoist [2].

Let us establish the opposite inclusion $\supset$. Assume the contrary, i.e., that there exists $p \in \partial f(x) \setminus \overline{\text{co}}^* L(f, x)$. Then by the separation theorem there exists $h \in S$ and $\delta > 0$ such that

$$f^0(x, h) \geq \langle p, h \rangle > \sup \langle L(f, x), h \rangle + \delta. \tag{5.1}$$

There exist sequences $x_n \xrightarrow[n \to \infty]{} x$, $t_n \downarrow 0$ such that

$$f^0(x; h) = \lim_{n \to \infty} \frac{f(x_n + t_n h) - f(x_n)}{t_n}.$$

By [2, Lemma 7] we have that f_λ and $(f_\lambda)^\mu$ are Lipschitz functions with the same Lipschitz constant L as the function f and

$$f_\lambda(x) = \inf_{u \in B(x; 2\lambda L)} \left\{ f(u) + \frac{1}{2\lambda} \|x - u\|^2 \right\},$$

so there exist $\lambda_n > 0$ and $y_n \in E$ such that

$$f_{\lambda_n}(x_n + t_n h) > f(y_n) + \frac{1}{2\lambda_n} \|x_n + t_n h - y_n\|^2 - t_n^2,$$

$$\|x_n + t_n h - y_n\| \leq 2L\lambda_n < t_n^2.$$

Using the same arguments we have that there exist $\mu_n \in (0, \lambda_n)$ and $z_n \in E$ such that

$$(f_{\lambda_n})^{\mu_n}(x_n) < f_{\lambda_n}(z_n) - \frac{1}{2\mu_n} \|x_n - z_n\|^2 + t_n^2,$$

$$\|x_n - z_n\| < 2L\mu_n < t_n^2.$$

Hence we have

$$\begin{aligned}
f^0(x; h) &= \lim_{n \to \infty} \frac{f(y_n) - f(y_n - t_n h)}{t_n} \\
&\leq \limsup_{n \to \infty} \frac{f_{\lambda_n}(x_n + t_n h) - f_{\lambda_n}(x_n)}{t_n} \\
&\leq \limsup_{n \to \infty} \frac{(f_{\lambda_n})^{\mu_n}(x_n + t_n h) - (f_{\lambda_n})^{\mu_n}(x_n)}{t_n} \\
&\leq \limsup_{n \to \infty} \langle p_n, h \rangle = \lim_{k \to \infty} \langle p_{n_k}, h \rangle \\
&\leq \langle p_0, h \rangle,
\end{aligned}$$

where $p_n \in \partial(f_{\lambda_n})^{\mu_n}(c_n)$, $c_n \in (x_n, x_n + t_n h)$ by the mean value theorem of Lebourg (see [5]) and p_0 is a w^* limit of a generalized subsequence of $\{p_{n_k}\}_{k=1}^{\infty}$, which is norm bounded by the uniform Lipschitz property of $(f_{\lambda_n})^{\mu_n}$. This is a contradiction and the proof is completed. ∎

Let us mention that in the above characterization, the sequential definition of $L(f, x)$ is not enough as in Theorems 4.2 and 4.4.

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